MATH 303 – Measures and Integration
Homework 11

Please upload a pdf of your solutions by 23:59 on Monday, December 9. The assignment will be graded out of 8 points. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. The goal of this problem is to show that almost everywhere convergence is a "non-topological" notion of convergence.

(a) Let X be a topological space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to $x \in X$ if and only if for every open neighborhood U of x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. Show that the following are equivalent:

   (i) $(x_n)_{n\in\mathbb{N}}$ converges to x

   (ii) every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(x_{n_{k_l}})_{l\in\mathbb{N}}$ that converges to x

(b) Let $(X, \mathcal{B}, \mu)$ be a measure space, and suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of integrable functions that converges in $L^1(\mu)$ to an integrable function f. Show that every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges μ-almost everywhere to f.

(c) Define the *typewriter sequence* $f_n : [0,1) \to \{0,1\}$ by

$$f_1 = 1,$$

$$f_2 = \mathbb{1}_{[0,\frac{1}{2})}, f_3 = \mathbb{1}_{[\frac{1}{2},1)}$$

$$f_4 = \mathbb{1}_{[0,\frac{1}{4})}, f_5 = \mathbb{1}_{[\frac{1}{4},\frac{1}{2})}, f_6 = \mathbb{1}_{[\frac{1}{2},\frac{3}{4})}, f_7 = \mathbb{1}_{[\frac{3}{4},1)},$$

$$f_8 = \mathbb{1}_{[0,\frac{1}{8})}, f_9 = \mathbb{1}_{[\frac{1}{8},\frac{1}{4})}, f_{10} = \mathbb{1}_{[\frac{1}{4},\frac{3}{8})}, f_{11} = \mathbb{1}_{[\frac{3}{8},\frac{1}{2})}, f_{12} = \mathbb{1}_{[\frac{1}{2},\frac{5}{8})}, f_{13} = \mathbb{1}_{[\frac{5}{8},\frac{3}{4})}, f_{14} = \mathbb{1}_{[\frac{3}{4},\frac{7}{8})}, f_{15} = \mathbb{1}_{[\frac{7}{8},1)}$$

The general term of the sequence is

$$f_n = \mathbb{1}_{\left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right)}$$

for $k = \lfloor \log_2(n) \rfloor$. Let λ be the Lebesgue measure on $[0,1)$. Show that $f_n \to 0$ in $L^1(\lambda)$, but $(f_n(x))_{n\in\mathbb{N}}$ does not converge for any $x \in [0,1)$.

(d) Conclude that there is no topology on the space of Lebesgue-measurable functions $f : [0,1) \to \mathbb{R}$ such that almost everywhere convergence with respect to Lebesgue measure agrees with convergence in the topology.

Solution: (a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$.

(i) $\Longrightarrow$ (ii). Suppose $x_n \to x$ as $n \to \infty$. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. We claim $x_{n_k} \to x$ as $k \to \infty$. Let U be an open neighborhood of x. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Then since $n_k \geq k$, we have $x_{n_k} \in U$ for $k \geq N$. Therefore, $x_{n_k} \to x$ as $k \to \infty$.

(ii) $\Longrightarrow$ (i). We will prove the contrapositive. Suppose (i) fails. Then there exists an open neighborhood U of x such that $S = \{n \in \mathbb{N} : x_n \notin U\}$ is infinite. Let $n_1 < n_2 < \dots$ be an enumeration of S. Let $k_1 < k_2 < \dots$ be arbitrary. We claim that $(x_{n_{k_l}})_{l\in\mathbb{N}}$ does not converge to x, so (ii) fails. Indeed, for every $l \in \mathbb{N}$, $x_{n_{k_l}} \notin U$, since $n_{k_l} \in S$ by construction.

(b) Let $(f_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(f_n)_{n\in\mathbb{N}}$. A subsequence of a convergent sequence (in a metric space) is convergent, so $f_{n_k} \to f$ in $L^1(\mu)$ as $k \to \infty$. In particular, $(f_{n_k})_{k\in\mathbb{N}}$ is Cauchy. By Theorem 8.13 from the lecture notes, there is a subsequence $(f_{n_{k_l}})_{l\in\mathbb{N}}$ and a function $g \in L^1(\mu)$ such that $f_{n_{k_l}} \to g$ a.e. and in $L^1(\mu)$. But $f_{n_{k_l}} \to f$ in $L^1(\mu)$, so $\|f - g\|_1 \leq \left\|f - f_{n_{k_l}}\right\|_1 + \left\|f_{n_{k_l}} - g\right\|_1 \to 0$. That is, $f = g$ a.e. Thus, $f_{n_{k_l}} \to f$ a.e. as desired.

(c) First let us check that $f_n \to 0$ in $L^1(\lambda)$. For $n \in \mathbb{N}$ and $k = \lfloor \log_2(n) \rfloor$, we have

$$\|f_n\|_1 = \lambda\left(\left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right)\right) = \frac{1}{2^k} = \frac{2}{2^{k+1}} \leq \frac{2}{n}.$$

Therefore, $\|f_n\|_1 \to 0$ as $n \to \infty$, so $f_n \to 0$ in $L^1(\lambda)$.

Let $x \in [0,1)$. For each $k \geq 0$, we have a partition of $[0,1)$ into intervals of length 2^{-k} by

$$[0,1) = \bigsqcup_{j=0}^{2^k-1} \underbrace{\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right)}_{I_{k,j}}.$$

Let $j_k(x) \in \{0, 1, \ldots, 2^k - 1\}$ so that $x \in I_{k,j_k(x)}$ for each $k \geq 0$. By definition, $f_{2^k+j} = \mathbb{1}_{I_{k,j}}$ for $k \geq 0$ and $0 \leq j \leq 2^k - 1$, so $f_{2^k+j_k(x)}(x) = 1$ and $f_{2^k+j}(x) = 0$ for $j \neq j_k(x)$. Therefore, $\limsup_{n\to\infty} f_n(x) = 1$ and $\liminf_{n\to\infty} f_n(x) = 0$, so $(f_n(x))_{n\in\mathbb{N}}$ does not converge.

(d) Suppose for contradiction that there is a topology τ on the space of measurable functions $f : [0,1) \to \mathbb{R}$ such that $f_n \to f$ with respect to τ if and only if $f_n \to f$ λ-a.e. Let f_n be the typewriter sequence from part (c). As shown in (c), $f_n \to 0$ in $L^1(\mu)$. Therefore, by (b), every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges to 0 a.e. and hence with respect to τ by assumption. But τ is a topology, so by the implication (ii) $\Longrightarrow$ (i) in (a), $f_n \to 0$ with respect to τ. That is, $f_n \to 0$ a.e. This contradicts the second part of (c), where we showed that $\{x \in [0,1) : f_n(x) \to 0\} = \emptyset$. Thus, there is no such topology τ.